lowing manner:—If the curve determined by the equation $V=0$ be traced in the first quadrant, every straight line parallel to the axis of $p$ meets the curve once, and only once, and every straight line parallel to the axis of $\tau$, and at a greater distance from it than $p_0$, meets the curve once, and only once. As $\frac{dV}{d\tau}$ vanishes with $\tau$, the curve meets the axis of $p$ at right angles; and from Art. 15 it follows that when $\tau$ and $p$ are indefinitely great, the angle which the tangent to the curve makes with the axis of $p$ is very nearly a right angle. Thus, for small values of $\tau$ the curve is concave to the axis of $p$, and for very large values of $\tau$ the curve is convex to the axis of $p$; so that the curve must have a point or points of inflexion.

24. In the very careful account of Ivory's mathematical researches, which is given in the fourth volume of the 'Abstracts of the Papers . . . of the Royal Society,' it is said, with respect to Jacobi's theorem, "In a paper in the Transactions for 1838, Mr. Ivory has with great elegance demonstrated this theorem, and has given, with greater detail than its authors had entered on, several statements regarding the limitations of the proportions of the axes." The language is cautious, but seems to imply some suspicion with regard to the accuracy of the statements. As we have now seen, many of Ivory's statements are inaccurate, and others, though accurate, are based on unsound reasoning.

## "On the Theory of Continuous Beams." By John Mortimer Heppel, M. Inst. C.E. Communicated by Prof. W. J. Macquorn Rankine. Received December 9, 1869*.

In venturing to present to the Royal Society a paper on a subject which has engaged the attention, more especially in France, of some of the most eminent engineers and writers on Mechanical Philosophy, the author feels it to be incumbent on him to state the nature of the claim to their attention which he hopes it may be found to possess in point of originality or improvement on the method of treatment.

To do this clearly, however, it will be necessary to advert to the principal steps by which progress in the knowledge of this subject has been made, both in France and in this country.

The theory of continuous beams appears to have first attracted attention in France about 1825, when a method of determining all the conditions of equilibrium of a straight beam of uniform section throughout, resting on any number of level supports at any distances apart, each span being loaded uniformly, but the uniform loads varying in any manner from one span to another, was investigated and published by M. Navier. This method, although perfectly exact for the assumed conditions, was objectionable from the great labour and intricacy of the calculations it entailed.

* Read January 27, 1870. See vol. xviii. p. 176.

Messrs. Molinos and Pronnier, in their work entitled 'Traité Théorique et Pratique de la construction des Ponts Métalliques,' describe this process fully, and show that for a bridge of $n$ openings, the solution must be effected of $3n+1$ equations, involving as many unknown quantities, these equations being themselves of a complex character; and they observe, "Thus to find the curve of the moments of rupture for a bridge of 6 spans 19 equations must be operated on; such calculations would be repulsive, and when the number of spans is at all considerable this method must be abandoned."

The method of M. Navier, however, remained the only one available till about 1849, when M. Clapeyron, Ingénieur des Mines, and Member of the Academy of Sciences, being charged with the construction of the Pont d'Asnières, a bridge of five continuous spans over the Seine, near Paris, applied himself to seek some more manageable process. He appears to have perceived (and so far as the writer is informed, to have been the first to perceive) that if the bending moments over the supports at the ends of any span were known as well as the amount and distribution of the load, the entire mechanical condition of this portion of the beam would become known just as if it were an independent beam. Upon this M. Clapeyron proceeded to form a set of equations involving as unknown quantities the bending moments over the supports, with a view to their determination. He found himself, however, obliged to introduce into these equations a second set of unknown quantities ("*inconnues auxiliaires*"), being the inclinations of the deflection curve at the points of support, and not having arrived at a general method of eliminating these latter, was obliged to operate in each case on a number of equations equal to twice the number of spans. M. Clapeyron does not appear, as yet, to have made any formal publication of his method, but to have used it in his own practice, and communicated it freely to those with whom he came in contact.

In 1856, M. Bertot, Ingénieur Civil, appears to have found the means of eliminating this second set of unknown quantities $n+1$ in number for a bridge of $n$ spans, and thus reducing the number of equations to $n-1$.

Each of these equations involved as unknown quantities the bending moments over three consecutive supports, and was considered, from its remarkable symmetry and simplicity, to merit a distinctive name, that of "The Theorem of the three Moments."

The method, however, to which this theorem is the key, is still everywhere called that of M. Clapeyron, and, as it appears to the writer, justly so, as it was an immediate and simple result from his investigations, with which M. Bertot was well acquainted.

The next important advance was made in 1861, when M. Bresse, Professeur de Mécanique appliquée à l'École Impériale des Ponts et Chaussées, completed the matter of the third volume of his course, which is exclusively devoted to this subject*. M. Bresse explains and de-

* This was communicated to the Academy of Sciences in 1862, though the volume was not published till 1865.

monstrates the theorem of the three moments, at the knowledge of which he had himself arrived from M. Clapeyron's investigations, independently of M. Bertot. He then goes on to the investigation of an equation of much greater generality, in which what is termed by English writers "imperfect continuity" is taken into account, being, however, there replaced by the precisely equivalent notion of original differences of level in the supports, the beam being always supposed primitively straight; besides this the loads, instead of being taken as uniform for each span, are considered as distributed in any given manner.

Having obtained this fundamental equation, M. Bresse proceeds to investigate the nature of the curves, which are the envelopes of the greatest bending moments produced at each point, by the most unfavourable distribution of the load in reference to it, and finally gives tables for the ready calculation of results in a great variety of cases, comprising most of those likely to occur in practice.

During the time that M. Bresse was engaged in these researches, an Imperial Commission was formed, of which he was a member, for the purpose of devising rules applicable to practice, and the results of his labours have been the basis of legislative enactments equivalent to our Board of Trade regulations prescribing the methods to be followed in determining the stresses in the various parts of the structure.

About the same time that M. Bresse turned his attention to this subject, it appears also to have engaged that of M. Bélanger, who in his work entitled 'Théorie de la Resistance et de la Flexion Plane des Solides &c., Paris, 1862,' gives a very complete demonstration—resulting in an equation which in one point of view is slightly more general than that of M. Bresse, as it takes in variation of the moment of inertia of the section from one span to another. In another point of view its generality is slightly less, as it deals only with loads distributed over each separate span uniformly, whereas M. Bresse replaces the simple algebraical terms expressing these by definite integrals expressing the load as a function of the distance from one of the points of support.

As far as the writer is informed, little has been done in France to advance this theory beyond the point to which it was brought by the writers last mentioned, and especially by M. Bresse; but valuable contributions to its development in reference to application to practice are to be found in the work of MM. Molinos and Pronnier above referred to, as well as in various papers by MM. Renaudot, Albaret, Colignon, Piarron de Mondesir, &c.

In England little or no attention appears to have been paid to this subject by writers on mechanics till 1843, when the Rev. Henry Moseley, Professor of Natural Philosophy and Astronomy at King's College, London, published his work on 'The Mechanical Principles of Engineering and Architecture.' In part 5 of this work, which treats of the strength of materials, four cases of continuous beams are fully investigated, and the

general case is to a certain extent discussed, the method of M. Navier being perhaps rather indicated than fully developed.

Prof. Moseley's work was altogether a most valuable contribution to engineering science, and, as far as the present subject is concerned, no doubt furnished the groundwork of the method applied by Mr. Pole to the solution of other particular but more complex and difficult cases.

The first case which engaged the attention of Mr. Pole appears to have been that of the bridge over the Trent at Torksey, consisting of two spans of continuous tubular beams, resting on abutments and a central pier. For special reasons it had become necessary that the real conditions of equilibrium of this bridge should be investigated with more than ordinary precision; and this Mr. Pole did by a method virtually identical with that of M. Navier, though it does not appear that he had any previous knowledge of that method, except through the medium of Moseley's work. Throughout Moseley's cases, however, the load on the beam is considered as distributed uniformly over its entire length, whereas Mr. Pole had to deal with the case of different loads on the two spans, and no doubt had to devise the method of analysis necessary for its treatment. Mr. Pole's paper on this subject is published in vol. ix. of the 'Minutes of Proceedings Inst. Civ. En.' 1849–50.

As far as this went, however, it could hardly be considered to have advanced the theory of the subject, as M. Navier's method included this case, and much more; but about the same time Mr. Pole had to investigate the case of a much larger work, the Britannia Bridge, where he had to deal with some new conditions, which, as far as the writer is aware, were then for the first time successfully treated.

These were that, besides variation of load on the different spans, their cross sections also varied, and there was imperfect continuity over the centre pier, that is to say, that the points of support being supposed to range in a straight line, the beam if relieved from all weight would cease to remain in contact with them all, and would consist of two equal straight portions, forming an angle pointing upwards. The process, which for distinction may be called that of M. Navier, was skilfully extended by Mr. Pole so as to include these new circumstances, and by its means results were obtained certainly true within a very small limit, and as near the absolute truth as any existing means of treating the subject would produce.

Mr. Pole's researches on this subject are published in Mr. Edwin Clark's work on the Britannia and Conway Bridges, 1850. Both from the clear and accurate treatment of the case and the record of the numerous and delicate observations by which the theoretical conclusions were continually verified and kept in check, they are most strongly to be recommended to the attention of engineers having to deal with works of this character.

The sequence of events now compels the writer to advert to some studies of his own. In 1858–59, being then Chief Engineer of the Madras Rail-

way, he had occasion to investigate the conditions of a bridge of five continuous spans over the River Palar. Having in India no books to refer to but those of Moseley and Edwin Clark, he found himself unable to extend the treatment of the cases there given to that of a beam with an increased number of openings and varying loads. After many attempts and failures, the same idea occurred to him which appears to have struck M. Clapeyron nine or ten years before, that if the bending moments over the supports were known, the whole conditions would become known.

Following this clue, he was fortunate enough to succeed in at once eliminating the other unknown quantities, which M. Clapeyron had been obliged to retain in his equations for many years after his original discovery of the method, and thus to arrive at an equation precisely identical with that which had been first published in France by M. Bertot in 1856, and was known as the "Theorem of the three Moments."

This was sufficient for the immediate purpose, as the beams in question were straight and of uniform section throughout, conditions to which this theorem is strictly applicable without any modification whatever.

As, however, the writer was at this time under the impression that he was using an entirely new mode of analysis, he was naturally anxious to check its results by comparison with those obtained in some well-known case by other means. Fortunately he had at hand that of the Britannia Bridge, perhaps the best that could have been selected ; but for this purpose it became necessary to import into the fundamental equation the conditions of varying sections in the different spans and imperfect continuity. This, however, presented no great difficulty, and by means of an equation thus modified, he had the satisfaction of reproducing all Mr. Pole's results, and thus convincing himself of the trustworthiness of the method in question.

The equation thus generalized is absolutely identical with that arrived at by M. Bélanger in the work above referred to*.

It would appear, then, that the theory of this subject was independently advanced to about the same state of perfection in France and in England, though as regards the development of its application to practice no doubt very much the more has been done in the former country.

The writer will now advert to some inherent defects of this theory, the cure of which is the principal object of the investigation which follows.

The chief one, which is admitted by all writers on the subject, is the necessity for supposing the moment of inertia of the section constant throughout each span ; any more general hypothesis, it is said, would render the calculation inextricable. Still it is certain that the conclusions arrived at on the hypothesis of a constant section cease to be true if a variation of section is introduced, and the amount of error thereby induced, though considered to be probably small, is still a matter of uncertainty.

The next defect is the assumption of uniformity of load throughout

* A paper on this subject by the writer was published in the Minutes of Proceedings Inst. C.E. vol. xix. 1859-60.

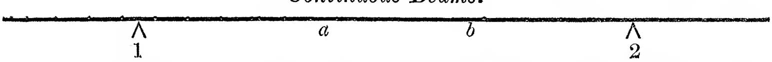
each span; for although as far as rolling load is concerned no more correct hypothesis could be made, the weight of the bridge itself, if a large one, usually varies considerably in the different parts of the same span.

The equation given by M. Bresse, as has been stated, provides for certain kinds of variable loads by the use of integrals, but the writer is not aware that they have been applied, even by that author himself, to the purposes of calculation, and it seems to him that in most cases the attempt to make such an application would be beset with difficulties.

It will, however, it is hoped, be seen from what follows, that the dealing with variations of the above elements does not in fact present any very formidable difficulty, though no doubt the labour of calculation is greater; but what the writer regards as most satisfactory is the very small difference in the principal results in the case of the Britannia (where these variations greatly exceed in amount those usually occurring), whether obtained by the approximate method hitherto followed, or by the more rigorous one to be explained, affording a strong presumption that in all ordinary cases the former method may be confidently employed without risk of any important error.

Should the following treatment of the case be deemed successful, the author would remark that its success is mainly due to the use of an abbreviated functional notation, by which a great degree of clearness and symmetry is preserved in expressions which would otherwise have become inextricably complex.

*General Investigation of the Bending Moments and Deflections of Continuous Beams.*




Let 1 . 2 represent any span of a continuous beam, the length of the span being $l$.

$x \, . \, y$ the coordinates of the deflection curve, the origin being at the point 1.

$a$ and $b$ particular values of $x$.

$\epsilon_1$, $\epsilon_2$, $\epsilon_3$ reciprocals of the products of the moments of inertia of the sections in the spaces $(1 \, . \, a)$, $(a \, . \, b)$, $(b \, . \, 2)$, about their neutral axes, by the modulus of elasticity of the material $\left(\frac{1}{\mathrm{EI}}\right)$.

$\mu_1$, $\mu_2$, $\mu_3$ loads per unit of length in the same spaces.

T tangent of inclination of deflection curve at 1, to straight line joining 1 . 2, its positive value being taken upwards.

$\phi_1$, $\phi_2$ bending moments at 1 . 2.

P shearing force at 1.

Now let the bending moment at any point $(x \, . \, y)$

between 1 and $a$ be called $F_1''(x)$,
between $a$ and $b$ be called $F_2''(x)$,
between $b$ and 2 be called $F_3''(x)$;

and let the part of this bending moment, which results alone from the load on the beam between 1 and $x$, be called

between 1 and $a$ $f_1''(x)$,
between $a$ and $b$ $f_2''(x)$,
between $b$ and 2 $f_3''(x)$;

and let the first and second integrals of these functions, as of $F_1''(x)$, $f_1''(x)$, be denoted by $F_1'(x)$, $f_1'(x)$, and $F_1(x)$, $f_1(x)$, and the value of any one, as $F_1(x)$, for a particular value of $x$, as $a$ by $F_1(a)$;

then $$f_1''(x)=\mu_1\frac{x^2}{2}, \qquad (1)$$

$$f_2''(x)=\mu_1 a\left(x-\frac{a}{2}\right)+\mu_2\frac{(x-a)^2}{2}\Big), \qquad (2)$$

$$f_3''(x)=\mu_1 a\left(x-\frac{a}{2}\right)+\mu_2(b-a)\left(x-\frac{b+a}{2}\right)+\mu_3\frac{(x-b)^2}{2}. \qquad (3)$$

Also, from equality of moments about the point $(x \,.\, y)$,

$$F_1''(x)=\phi_1-Px+f_1''(x), \qquad (4)$$

$$F_2''(x)=\phi_1-Px+f_2''(x), \qquad (5)$$

$$F_3''(x)=\phi_1-Px+f_3''(x); \qquad (6)$$

and, from equality of moments about the point 2,

$$Pl=\phi_1-\phi_2+f_3''(l),$$

$$P=\frac{1}{l}\left(\phi_1-\phi_2+f_3''(l)\right). \qquad (7)$$

Substituting for P in (4), (5) and (6),

$$F_1''(x)=\left(1-\frac{x}{l}\right)\phi_1+\frac{x}{l}\phi_2-\frac{x}{l}f_3''(l)+f_1''(x), \qquad (8)$$

$$F_2''(x)=\left(1-\frac{x}{l}\right)\phi_1+\frac{x}{l}\phi_2-\frac{x}{l}f_3''(l)+f_2''(x), \qquad (9)$$

$$F_3''(x)=\left(1-\frac{x}{l}\right)\phi_1+\frac{x}{l}\phi_2-\frac{x}{l}f_3''(l)+f_3''(x), \qquad (10)$$

equations from which for a given value of $x$, $F_1''(x)$, $F_2''(x)$, $F_3''(x)$ may be determined if $\phi_1$ and $\phi_2$ are known.

From the nature of the deflection curve,

from 1 to $a$,

$$\frac{d^2y}{dx^2}=\epsilon_1 F_1''(x); \qquad (11)$$

from $a$ to $b$,

$$\frac{d^2y}{dx^2}=\epsilon_2 F_2''(x); \qquad (12)$$

from $b$ to 2,

$$\frac{d^2y}{dx^2}=\epsilon_3 F_3''(x); \quad \ldots \quad (13)$$

$\therefore$ from 1 to $a$,

$$\frac{dy}{dx}=\epsilon_1 F'_1(x)+C, \text{ when } x=0, \; F_1'(x)=0, \; \frac{dy}{dx}=-T;$$

$$\therefore \frac{dy}{dx}=\epsilon_1 F_1'(x)-T; \quad \ldots \quad (14)$$

from $a$ to $b$,

$$\frac{dy}{dx}=\epsilon_2 F_2'(x)+C; \quad \ldots \quad (15)$$

making $x=a$ in (14) and (15), and transposing,

$$C=\epsilon_1 F_1'(a)-\epsilon_2 F_2'(b)-T;$$

$$\therefore \frac{dy}{dx}=\epsilon_1 F_1'(a)-\epsilon_2\Big(F_2'(x)-F_2'(a)\Big)-T; \quad \ldots \quad (16)$$

from $b$ to 2,

$$\frac{dy}{dx}=\epsilon_3 F_3'(x)+C; \quad \ldots \quad (17)$$

making $x=b$ in (16) and (17), and transposing,

$$C=\epsilon_1 F_1'(a)+\epsilon_2\Big(F_2'(b)-F_2(a)\Big)-\epsilon_3 F'_3(b);$$

$$\therefore \frac{dy}{dx}=\epsilon_1 F_1'(a)+\epsilon_2\Big(F_2'(b)-F_2'(a)\Big)+\epsilon_3\Big(F_3'(x)-F_3'b\Big)-T; \quad . \ (18)$$

$\therefore$ from 1 to $a$,

$$y=\epsilon_1 F_1(x)-Tx, \text{ no constant; for if } x=0, \; F_1 x=0, \; y=0; \quad \ldots \quad (19)$$

from $a$ to $b$,

$$y=\epsilon_1 F_1'(a)x+\epsilon_2\big(F_2(x)-F_2'(a)x\big)-Tx+C; \quad \ldots \quad (20)$$

making $x=a$ in (19) and (20), and transposing,

$$C=\epsilon_1\big(F_1 a-F_1'(a)a\big)+\epsilon_2(F_2(a)-F'_2(a)a);$$

$$\therefore y=\epsilon_1\big(F_1(a)+F_1'(a)(x-a)\big)+\epsilon_2\Big(F_2(x)-\big(F_2(a)+F_2'(a)(x-a)\big)\Big)-Tx; \quad \ldots \quad (21)$$

from $b$ to 2,

$$y=\epsilon_1 F_1'(a)x+\epsilon_2\big(F_2'(b)x-F_2'(a)x\big)+\epsilon_3\big(F_3(x)-F'_3(b)x\big)-Tx+C; \quad \ldots \quad (22)$$

making $x=b$ in (21) and (22), and transposing,

$$y=\epsilon_1\big(F_1(a)+F_1'(a)(x-a)\big)+\epsilon_2\Big[\big(F_2(b)+F_2'(b)(x-b)\big)-\big(F_2(a)+F_2'(a)(x-a)\big)\Big]$$
$$+\epsilon_3\Big[F_3(x)-\big(F_3(b)+F_3'(b)(x-b)\big)\Big]-Tx \quad \ldots \quad (23)$$

From the way in which this last equation is formed, it is evident that if

there were any number of particular values of $x$ to be considered, as $a \,.\, b$ &c., $j$, $k$, $l$, the corresponding values of $\frac{1}{\mathrm{EI}}$ being $\epsilon_1$, $\epsilon_2$, &c., $\epsilon_{n-1}$, $\epsilon_n$, it might be written

$$y=\left\{\begin{array}{l} \epsilon_1\big(\mathrm{F}_1(a)+\mathrm{F}_1'(a)(x-a)\big) \\ +\epsilon_2\Big[\big(\mathrm{F}_2(b)+\mathrm{F}_2'(b)(x-b)\big)-\big(\mathrm{F}_2(a)+\mathrm{F}_2'(a)(x-a)\big)\Big] \\ +\epsilon_3\Big[\big(\mathrm{F}_3(c)+\mathrm{F}_3'(c)(x-c)\big)-\big(\mathrm{F}_3(b)+\mathrm{F}_3(b)(x-b)\big)\Big] \\ +\ \&c. \\ +\epsilon_{n-1}\Big[\big(\mathrm{F}_{n-1}(k)+\mathrm{F}'_{n-1}(k)(x-k)\big)-\big(\mathrm{F}_{n-1}(j)+\mathrm{F}'_{n-1}(j)(x-j)\big)\Big] \\ +\epsilon_n\Big[\mathrm{F}_n(x)-\big(\mathrm{F}_n(k)+\mathrm{F}'_n(k)(x-k)\big)\Big]\,; \end{array}\right\} -\mathrm{T}x\,; \quad (24)$$

if $x=l$ in (24), $y=0$;

$$\therefore \mathrm{T}=\frac{1}{l}\left\{\begin{array}{l} \epsilon_1\Big[\mathrm{F}_1(a)+\mathrm{F}_1'(a)(l-a)\Big] \\ +\epsilon_2\Big[\big(\mathrm{F}_2(b)+\mathrm{F}_2'(b)(l-b)\big)-\big(\mathrm{F}_2(a)+\mathrm{F}_2'(a)(l-a)\big)\Big] \\ +\epsilon_3\Big[\big(\mathrm{F}_3(c)+\mathrm{F}_3'(c)(l-c)\big)-\big(\mathrm{F}_3(b)+\mathrm{F}'_3(b)(l-b)\big)\Big] \\ +\ \&c. \\ +\epsilon_{n-1}\Big[\big(\mathrm{F}_{n-1}(k)+\mathrm{F}'_{n-1}(k)(l-k)\big)-\big(\mathrm{F}_{n-1}(j)+\mathrm{F}'_{n-1}(j)(l-j)\big)\Big] \\ +\epsilon_n\Big[\mathrm{F}_n(l)-\big(\mathrm{F}_n(k)+\mathrm{F}'_n(k)(l-k)\big)\Big]. \end{array}\right\} \quad \ldots \ (25)$$

If, now, the formation of the functions $\mathrm{F}_1(a)$, $\mathrm{F}_1'(a)$ &c. be examined, it is evident that this equation may be written

$$\mathrm{T}=\mathrm{A}\phi_1+\mathrm{B}\phi_2+\mathrm{C},$$

where A and B are known functions of $a$, $b$, $c$, &c., and $\epsilon_1$, $\epsilon_2$, $\epsilon_3$, &c., and C is a known function of the same, and $\mu_1$, $\mu_2$, $\mu_3$, &c.

If the adjacent span to the left be now considered, it is evident that a precisely similar equation may be obtained, which may be written

$$\mathrm{T}'=\mathrm{A}'\phi_1+\mathrm{B}'\phi_0+\mathrm{C}'\,;$$

adding these, and writing $t$ for $\mathrm{T}+\mathrm{T}'$, which is known as it is the tangent of the small angle which the neutral lines of the two spans would make at the point 1 if relieved from all load,

$$t=(\mathrm{A}+\mathrm{A}')\phi_1+\mathrm{B}\phi_2+\mathrm{B}'\phi_0+\mathrm{C}+\mathrm{C}',$$

which may be written

$$\Psi_1(\phi_0,\ \phi_1,\ \phi_2)=0\,;$$

similarly for the other bearing points in succession,

$$\Psi_2(\phi_1,\ \phi_2,\ \phi_3)=0,$$
$$\Psi_3(\phi_2,\ \phi_3,\ \phi_4)=0,\ \&c.,$$

where the number of equations is two less than that of the quantities

$\phi_0$, $\phi_1$, &c., so that if two of these are known the rest may be determined. But the first and last are always known, being usually each $=0$. Therefore they may all be determined.

This being so, the bending moment at any point $(x \,.\, y)$ may be found from equations (8), (9), (10) and others of the same form; and the deflection may be found from equations (19), (21), (23), and others of the same form, regard being had to the interval of the beam in which the point under examination lies.

If, now, we suppose that $a=b=c=$ &c. $=l$, equation (25) reduces to

$$\mathrm{T}=\frac{1}{\mathrm{EI}}\left(\mathrm{F}_1(l)\right);$$

similarly,

$$\mathrm{T}'=\frac{1}{\mathrm{EI}'}\left(\mathrm{F}_1(l')\right);$$

$$\therefore\ t=\frac{1}{\mathrm{EI}}\mathrm{F}_1(l)+\frac{1}{\mathrm{EI}'}\mathrm{F}_1(l'),$$

$$\mathrm{EI}t=\mathrm{F}_1(l)+i(\mathrm{F}_1 l'),\quad \text{writing } i \text{ for } \frac{\mathrm{I}}{\mathrm{I}'},$$

$$=\left(\frac{l}{3}+\frac{il'}{3}\right)\phi_1+\frac{l}{6}\phi_2+\frac{il}{6}\phi_0-\frac{l^3}{24}\mu-\frac{il'^3}{24}\mu'.$$

Clearing of fractions and transposing,

$$8(l+il')\phi_1+4l\phi_2+4il'\phi_0=l^3\mu+il'^3\mu'+24\mathrm{EI}t, \quad \ldots \quad (26)$$

an equation which was given by the author in his paper before referred to, and which is nearly identical with the general equation of M. Bresse, and allowing for difference of notation precisely so with that of M. Bélanger.

If $i=1$ and $t=0$, which is the case of a straight beam of uniform section throughout,

$$8(l+l')\phi_1+4l\phi_2+4l'\phi_0=l^3\mu+l'^3\mu', \quad \ldots\ldots \quad (27)$$

which is the equation generally known as the theorem of the three moments.

If in equation (25) we put $l=a$, it becomes

$$\mathrm{T}=a\epsilon_1\left(\frac{1}{3}\phi_1+\frac{1}{6}\phi_2-\frac{1}{24}a^2\mu_1\right); \quad \ldots\ldots \quad (28)$$

and for the central deflection equation (19) becomes

$$\mathrm{Y}=a^2\epsilon_1\left(-\frac{1}{16}(\phi_1+\phi_2)+\frac{5}{384}a^2\mu_1\right). \quad \ldots \quad (29)$$

If we put $b=2a$, $l=3a$,

$$\mathrm{T}=a\left[\begin{array}{l}\epsilon_1\left(\frac{19}{27}\phi_1+\frac{7}{54}\phi_2-a^2\left(\frac{43}{216}\mu_1+\frac{7}{36}\mu_2+\frac{7}{108}\mu_3\right)\right)\\ +\epsilon_2\left(\frac{7}{27}\phi_1+\frac{13}{54}\phi_2-a^2\left(\frac{7}{54}\mu_1+\frac{7}{24}\mu_2+\frac{13}{108}\mu_3\right)\right)\\ +\epsilon_3\left(\frac{1}{27}\phi_1+\frac{7}{54}\phi_2-a^2\left(\frac{1}{54}\mu_1+\frac{1}{18}\mu_2+\frac{11}{216}\mu_3\right)\right)\end{array}\right], \quad (30)$$

and central deflection from equation (21),

$$\mathrm{Y}=a^2\left[\begin{aligned}&\epsilon_1\left(-\frac{7}{36}\phi_1-\frac{1}{18}\phi_2+a^2\left(\frac{11}{144}\mu_1+\frac{1}{12}\mu_2+\frac{1}{36}\mu_3\right)\right)\\&+\epsilon_2\left(-\frac{5}{16}\phi_1-\frac{5}{16}\phi_2+a^2\left(\frac{5}{32}\mu_1+\frac{47}{128}\mu_2+\frac{5}{32}\mu_3\right)\right)\\&+\epsilon_3\left(-\frac{1}{18}\phi_1-\frac{7}{36}\phi_2+a^2\left(\frac{1}{36}\mu_1+\frac{1}{12}\mu_2+\frac{11}{144}\mu_3\right)\right)\end{aligned}\right]. \tag{31}$$

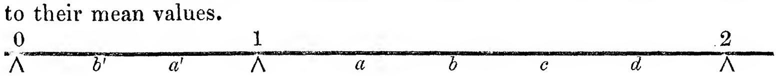
If we put $b=2a,\ c=3a,\ d=4a,\ l=5a,$

$$\mathrm{Y}=a\left[\begin{aligned}&\epsilon_1\left(\frac{61}{75}\phi_1+\frac{13}{150}\phi_2-a^2\left(\frac{149}{600}\mu_1+\frac{91}{300}\mu_2+\frac{13}{60}\mu_3+\frac{13}{100}\mu_4+\frac{13}{300}\mu_5\right)\right)\\&+\epsilon_2\left(\frac{37}{75}\phi_1+\frac{31}{150}\phi_2-a^2\left(\frac{37}{150}\mu_1+\frac{123}{200}\mu_2+\frac{31}{60}\mu_3+\frac{31}{100}\mu_4+\frac{31}{300}\mu_5\right)\right)\\&+\epsilon_3\left(\frac{19}{75}\phi_1+\frac{37}{150}\phi_2-a^2\left(\frac{19}{150}\mu_1+\frac{19}{50}\mu_2+\frac{13}{24}\mu_3+\frac{37}{100}\mu_4+\frac{37}{300}\mu_5\right)\right)\\&+\epsilon_4\left(\frac{7}{75}\phi_1+\frac{31}{150}\phi_2-a^2\left(\frac{7}{150}\mu_1+\frac{7}{50}\mu_2+\frac{7}{30}\mu_3+\frac{161}{600}\mu_4+\frac{31}{300}\mu_5\right)\right)\\&+\epsilon_5\left(\frac{1}{75}\phi_1+\frac{13}{150}\phi_2-a^2\left(\frac{1}{150}\mu_1+\frac{1}{50}\mu_2+\frac{1}{30}\mu_3+\frac{7}{150}\mu_4+\frac{7}{200}\mu_5\right)\right)\end{aligned}\right], \tag{32}$$

and central deflection from equation (23),

$$\mathrm{Y}=a^2\left[\begin{aligned}&\epsilon_1\left(-\frac{13}{60}\phi_1-\frac{1}{30}\phi_2+a^2\left(\frac{7}{80}\mu_1+\frac{7}{60}\mu_2+\frac{1}{12}\mu_3+\frac{1}{20}\mu_4+\frac{1}{60}\mu_5\right)\right)\\&+\epsilon_2\left(-\frac{31}{60}\phi_1-\frac{7}{30}\phi_2+a^2\left(\frac{31}{120}\mu_1+\frac{161}{240}\mu_2+\frac{7}{12}\mu_3+\frac{7}{20}\mu_4+\frac{7}{60}\mu_5\right)\right)\\&+\epsilon_3\left(-\frac{9}{16}\phi_1-\frac{9}{16}\phi_2+a^2\left(\frac{9}{32}\mu_1+\frac{27}{32}\mu_2+\frac{469}{384}\mu_3+\frac{27}{32}\mu_4+\frac{9}{32}\mu_5\right)\right)\\&+\epsilon_4\left(-\frac{7}{30}\phi_1-\frac{31}{60}\phi_2+a^2\left(\frac{7}{60}\mu_1+\frac{7}{20}\mu_2+\frac{7}{12}\mu_3+\frac{161}{240}\mu_4+\frac{31}{120}\mu_5\right)\right)\\&+\epsilon_5\left(-\frac{1}{30}\phi_1-\frac{13}{60}\phi_2+a^2\left(\frac{1}{60}\mu_1+\frac{1}{20}\mu_2+\frac{1}{12}\mu_3+\frac{7}{60}\mu_4+\frac{7}{80}\mu_5\right)\right)\end{aligned}\right]. \tag{33}$$

As an example of the application of the foregoing method to the purposes of calculation, let the case of the Britannia Bridge be taken, and let the large span be supposed to be divided into five, and the small span into three equal parts, and let the moments of inertia of the sections and loads per unit of length be supposed constant within each part and equal to their mean values.



We have then the following data:—

In spans (1 . 2) and (1 . 0),

$$a=92, \quad b=2a, \quad c=3a, \quad d=4a, \quad l=5a,$$
$$a'=76\cdot7, \quad b=2a, \quad l'=3a\,;$$
$$\epsilon_1=\frac{1}{1132\mathrm{E}}, \quad \epsilon_2=\frac{1}{1520\mathrm{E}}, \quad \epsilon_3=\frac{1}{1746\mathrm{E}}, \quad \epsilon_4=\frac{1}{1664\mathrm{E}}, \quad \epsilon_5=\frac{1}{1857\mathrm{E}},$$
$$\epsilon_1'=\frac{1}{1100\mathrm{E}}, \quad \epsilon_2'=\frac{1}{960\mathrm{E}}, \quad \epsilon_3=\frac{1}{720\mathrm{E}}\,;$$
$$\mu_1=2\cdot89, \quad \mu_2=3\cdot31, \quad \mu_3=3\cdot57, \quad \mu_4=3\cdot49, \quad \mu_5=3\cdot65,$$
$$\mu_1'=2\cdot84, \quad \mu_2'=2\cdot67, \quad \mu_3'=2\cdot32\,;$$
$$\mathrm{T}+\mathrm{T}'=0, \qquad \mathrm{E}=1440000.$$

In span (2 . 1),

$$a=92, \quad b=2a, \quad c=3a, \quad d=4a, \quad l=5a\,;$$
$$\epsilon_1=\frac{1}{1857\mathrm{E}}, \quad \epsilon_2=\frac{1}{1664\mathrm{E}}, \quad \epsilon_3=\frac{1}{1746\mathrm{E}}, \quad \epsilon_4=\frac{1}{1520\mathrm{E}}, \quad \epsilon_5=\frac{1}{1132\mathrm{E}}\,;$$
$$\mu_1=3\cdot65, \quad \mu_2=3\cdot49, \quad \mu_3=3\cdot57, \quad \mu_4=3\cdot31, \quad \mu_5=2\cdot89\,;$$

and from symmetry of loading $\mathrm{T}=\frac{1}{2}t=-0\cdot002035$.

Applying equation (30) and (32) to spans (1 . 0) and (1 . 2) respectively, and eliminating T and T′ by adding them, we obtain

$$0\cdot1888\phi_1+0\cdot04827\phi_2-10481=0\,;$$

and applying equation 32 to span (2 . 1),

$$0\cdot04827\phi_1+0\cdot08765\phi_2-5420=0,$$

whence $\qquad \phi_1=46206, \quad \phi_2=36387.$

Taking these values of $\phi_1$ and $\phi_2$, and applying equation (33) to the calculation of the deflection at the middle of the large span,

$$\mathrm{Y}=0\cdot375 \text{ ft.}=4\cdot5 \text{ inches.}$$

If, now, the values of $\phi_1$, $\phi_2$, and Y be calculated from equations (26) and (19), on the supposition that the moments of inertia of the section and the loads are constant throughout each span and equal to their mean values, they are

$$\phi_1=47030, \quad \phi_2=35610, \quad \mathrm{Y}=4\cdot62,$$

which are almost identical with the values ascertained by Mr. Pole.

If the variation of section alone be considered, the load being taken at its mean value,

$$\phi_1=46382, \quad \phi_2=34465, \quad \mathrm{Y}=4\cdot52.$$

It therefore appears that the amount of variation in the section and load which occurs in each span of the Britannia Bridge, when taken strictly into account, produces scarcely any effect on the values of the bending moments and deflections, which are practically the same as those resulting from their mean values considered as constant; and it may be

considered as demonstrated that, for most ordinary cases of large bridges, calculations founded on equation (26) may be confidently relied on. It need scarcely be remarked that these are much more simple and easy than those founded on the more exact but complex equations above given.

In smaller bridges, however, the error of the approximate process will be more considerable, and the process above given may be applied with advantage to its correction.

In concluding this paper, the author desires to record his thanks to his young friend, Mr. Henry Reilly, for the patience and skill with which he made, in detail, all the intricate calculations of the numerical values of the various functions involved in the above demonstration.

## "Remarks on Mr. Heppel's Theory of Continuous Beams." By W. J. Macquorn Rankine, C.E., LL.D., F.R.S. Received December 22, 1869*.

1. *Condensed form of stating the Theory.*—The advantages possessed by Mr. Heppel's method of treating the mathematical problem of the state of stress in a continuous beam will probably cause it to be used both in practice and in scientific study.

The manner in which the theory is set forth in Mr. Heppel's paper is remarkably clear and satisfactory, especially as the several steps of the algebraical investigation correspond closely with the steps of the arithmetical calculations which will have to be performed in applying the method to practice.

Still it appears to me that, for the scientific study of the principles of the method, and for the instruction of students in engineering science, it may be desirable to have those principles expressed in a condensed form; and with that view I have drawn up the following statement of them, which is virtually not a new investigation, but Mr. Heppel's investigation abridged.

Let $(x=0,\ y=0)$ and $(x=l,\ y=0)$ be the coordinates of two adjacent points of support of a continuous beam, $x$ being horizontal. Let $y$ and the vertical forces be positive downwards.

At a given point $x$ in the span between those points let $\mu$ be the load per unit of span, and EI the stiffness of the cross-section, each of which functions may be uniform or variable, continuous or discontinuous.

In each of the following double and quadruple definite integrals, let the lower limits be $x=0$.

$$\left.\begin{array}{ll}\displaystyle\iint \mu dx^2 = m\,; & \displaystyle\iint \frac{dx^2}{\mathrm{EI}} = n\,; \\ \displaystyle\iint \frac{x dx^2}{\mathrm{EI}} = q\,; & \displaystyle\iint \frac{dx^2}{\mathrm{EI}} \iint \mu dx^2 = \mathrm{F}. \end{array}\right\} \quad \ldots\ldots \quad (1)$$

When the integrations extend over the whole span $l$, that will be denoted by affixing 1; for example, $m_1$, $n_1$, &c.

* Read January 27, 1870. See vol. xviii. p. 178.